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Technical note

Transmission ratio based analysis and robust design of mechanisms

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ABSTRACT

This paper proposes an analytical approach to the robust design of mechanisms, to achieve motion and accuracy requirements given a desired transmission ratio and allowable geometrical variations. The focus is on four-bar and slider-crank mechanisms, which are common elements for the transmission of rotary motion, especially over distances, which are too big for the use of conventional elements such as gears, and motion along a predefined guide-curve, which often is a straight line. For many power transmission applications, a constant relation between the motions of an input and corresponding output element is required. For a four-bar linkage, a value of 1 is the only possible constant transmission ratio—achieved when the mechanism has a parallelogram configuration. In the case of a slider-crank linkage a constant transmission ratio can be achieved with a properly designed circular guide-curve, which makes the slider-crank essentially equivalent to a four-bar. In practice, however, as a result of variations in link lengths due to manufacturing tolerances and load-induced or thermal deformations, the transmission ratio for a parallelogram four-bar linkage will deviate substantially from its ideal theoretical value of 1. Even small changes in link lengths due to deformations or joint backlash can cause unacceptable instantaneous transmission ratio variations. The concepts presented are not limited to the design of four-bars and slider-cranks but can also be applied universally in the design of other mechanisms.

1. Introduction

When cost or space constraints restrict the use of multiple actuators and feedback control, a one degree of freedom/single input mechanism can be used to transfer power from a single actuator to two or more output components of a machine system. If synchronous motion between the output components is required then this necessitates a transmission ratio of 1 between them.

When transmitting power over distances that are too large for conventional transmission elements such as gears, a four link mechanism offers a simple means of power transmission. Using hinged or prismatic joints, four basic types of four link mechanisms are possible; the four-bar linkage, slider-crank, elliptic trammel and rapson slide [1]. All four types are shown in Fig. 1. Each of the mechanisms has a single degree of freedom and they are fully described by a single input with the output defined by some trigonometric function of the input angle and the link lengths.

In order to transmit synchronous motion, a parallelogram four-bar linkage can be used—for instance in the design of parallel mechanisms with two to six degrees of freedom [2]. A schematic of a generic four-bar linkage is shown in Fig. 2 with link lengths labeled. For the parallelogram configurations the input and output links will have the same lengths (i.e. $a=c$), as will the rocker and fixed links (i.e. $b=d$). In this configuration the input angle, θ_1, will always equal the output angle, θ_2.

However, any asymmetry in the mechanism will create singularities and therefore a configuration dependent transmission ratio resulting in errors in position and velocity between the ideally synchronized components. Furthermore, given that a parallelogram four-bar linkage is a borderline Grashof four-bar linkage, any variations in link lengths prevents coupled 360° rotation of the input and output links. Specifically, the output link will only be able to oscillate over an angular range of less than 180°. As mentioned, a perfect parallelogram four-bar linkage allows for continuous, synchronous rotation of the input and output links.

Deviation of link lengths from their ideal values can occur due to manufacturing tolerances and thermal or load-induced deformations. Any manufacturing process is associated with a dimensional tolerance that leads to variations in part dimensions and generally, the cost of manufacturing increases exponentially for closer tolerances [3]. Load-induced and mechanism position dependent deflection of the structure and bearings can also cause effective link length errors. These deflections are often distributed throughout the structure and a method for lumping them at discrete points must be devised (typically the bearing interface as it is often the most compliant part of the structure) [4]. The stiffness of the individual components can be computed by means of beam theory for simple parts or finite element analysis for parts with more compli-

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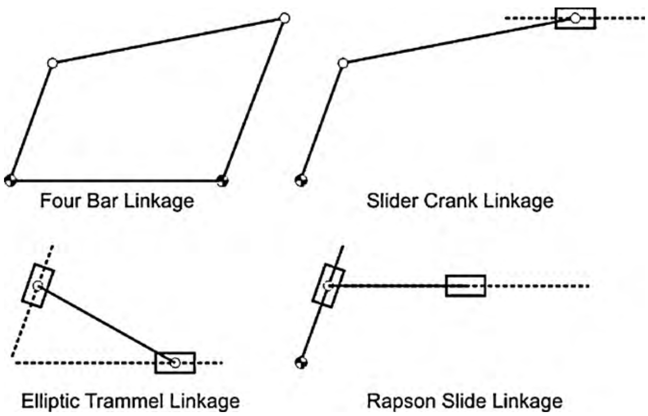

Fig. 1. Four possible types of four link mechanisms: (a) four-bar linkage: all four joints as revolute joints, (b) slider-crank mechanism: two revolute and one prismatic joint, (c) elliptical trammel linkage: two revolute and two prismatic joints on the same links. The name stems from the fact that the paths of all points on the coupler are ellipses, and (d) rapson slide linkage: this has two revolute joints and two prismatic joints but they are not on the same links.

cated geometry and then with knowledge of the forces in the links, estimates of the deflections can be obtained.

In order to design different kinds of mechanisms and predict their performance various kinematic techniques have been proposed. Fogarasy and Smith showed how mechanism constraint equations could be used in the tolerance analysis of mechanisms to predict the influence of manufacturing tolerances on the kinematic performance of mechanisms [5]. Hartenberg and Denavit differentiated the displacement equation of a four-bar function generator to derive the expression of the output sensitivity to link length perturbation [6]. Faik and Erdman performed a dimensional analysis to establish the relationship between the proportions of a mechanism and its sensitivity [7]. Furthermore the transmission angle (see Fig. 2) has been proposed as a means to evaluate the quality of motion transmission in a mechanism [8]. It is defined as the smaller angle between the direction of the velocity difference vector of the driving link and the direction of the absolute velocity vector of the output link both taken at the point of connection (this is the angle between the coupler and output link for the four-bar linkage). When the transmission angle is close to 90° a mechanism has the most effective force transmission and a low sensitivity to manufacturing tolerances of link lengths [9].

In this paper we show how the mechanism transmission ratio, derived from the kinematic constraint equation, can be a useful quantity in the design and analysis of four-bar linkages and provide a brief case study for a synchronous motion machine system.

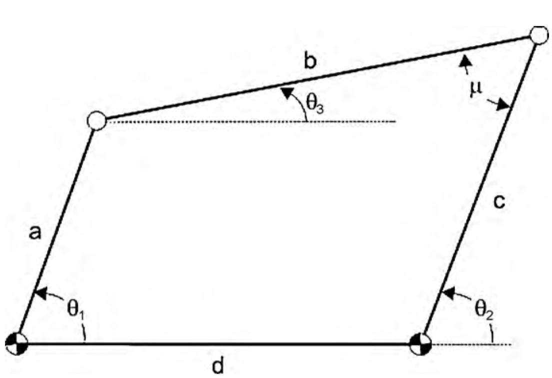

Fig. 2. Four-bar linkage notation. The transmission angle, μ, is defined as the angle between the follower link, b, and the coupler, c, of the four-bar linkage.

2. Transmission ratio of single degree of freedom planar mechanisms

For planar mechanisms, each link has three degrees of freedom; two translational and one rotational. When links are joined together, constraint relations reduce the number of generalized coordinates needed to describe the system. The mobility, m, of a mechanism (or its degrees of freedom) can be calculated using the Kutzbach–Gruebler equation:

$$m = 3(n-1) - 2j \quad (1)$$

where n is the number of links and j is the number of total joints. For planar mechanisms each joint can either be a single degree of freedom revolute or prismatic joint. For the case that a mechanism has a mobility of one, we typically define an input link and an output link. The configuration of a single degree of freedom mechanism can be described by one generalized coordinate, q.

2.1. Kinematic constraint equation

The kinematic constraint equation is determined from the geometry of the mechanism. It is based on the fact that there are two different but equivalent paths connecting two points on the same vector loop. For any single degree of freedom mechanism the general form of the constraint equation is some function, f, of the generalized coordinate q.

$$f(q) = 0 \quad (2)$$

2.2. Transmission ratio

The mechanism transmission ratio can be found by taking the derivative of the constraint equation with respect to the generalized coordinate.

$$\frac{df}{dq} = 0 \quad (3)$$

The transmission ratio relates the incremental changes in position (velocity) of the generalized coordinate to incremental changes in the output variables of the mechanism. With q_{out} being an output variable of the mechanism, the transmission ratio is given by:

$$r = \frac{\delta q_{out}}{\delta q} \quad (4)$$

3. Four-bar linkage

A generic four-bar linkage was shown in Fig. 2. Using the Kutzbach–Gruebler equation we can demonstrate that this mechanism has a mobility of one with Eq. (5) as the kinematic constraint equation.

$$a^2 - b^2 + c^2 + d^2 - 2ad\ \cos(\theta_1) + 2cd\cos(\theta_2) - 2ac\cos(\theta_1 - \theta_2) = 0 \quad (5)$$

The output angle, θ_2, can be found as a function of link lengths and input angle, θ_1, by solving:

$$A_1 + A_2\cos(\theta_2) + A_3\sin(\theta_2) = 0 \quad (6)$$

where

$$A_1 = a^2 - b^2 + c^2 + d^2 - 2ad\ \cos(\theta_1) \quad (7)$$

$$A_2 = 2cd - 2ac\ \cos(\theta_1) \quad (8)$$

$$A_3 = -2ac\sin(\theta_1) \quad (9)$$

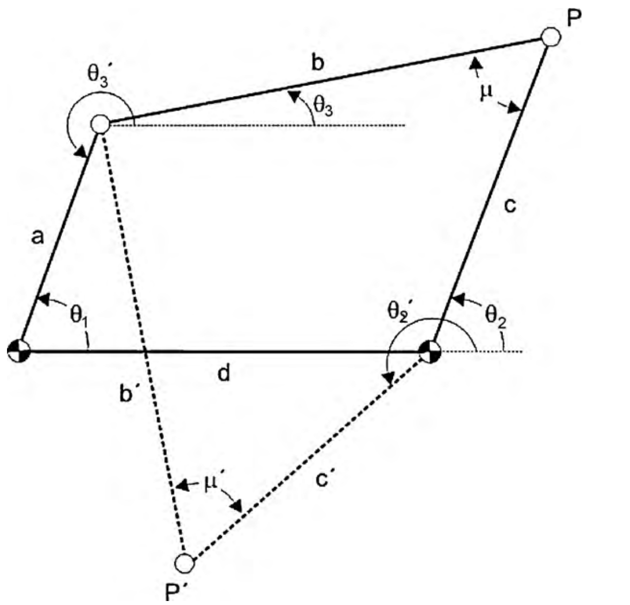

Fig. 3. The two possible positions (P and P′) of the point P for a given value of θ_2. There are two different possible values of θ_3 and two different values of θ_4 corresponding to the two possible positions of point P.

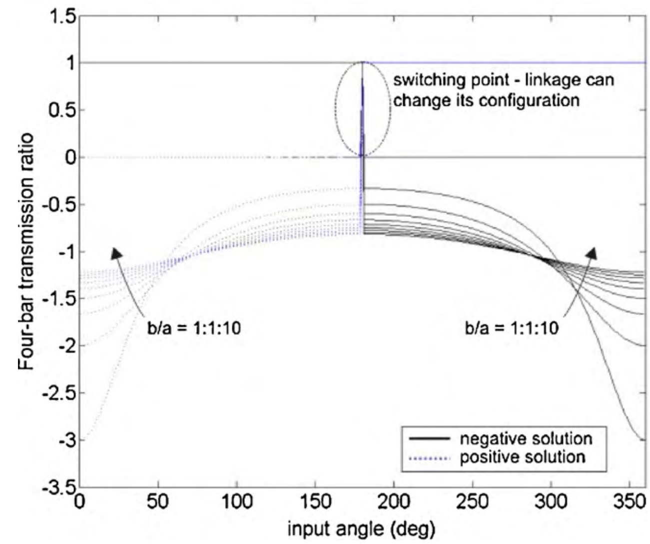

Fig. 4. Transmission ratios of the four-bar linkage when $a = c$ and $b = d$. The crossed configuration approaches −1 as the ratio b/a tends to infinity. Note the switching points at 0°, 180°, . . ., $n \times 180°$.

There are two solutions to Eq. (6) corresponding to the two possible configurations of the four-bar linkage that are shown in Fig. 3.

$$\tan\left(\frac{\theta_2}{2}\right) = \frac{-A_3 \pm \sqrt{A_3^2 - A_1^2 + A_2^2}}{A_1 - A_2} \tag{10}$$

The negative solutions corresponds to the parallelogram configuration for $0 < \theta_1 < \pi$ and to the crossed configuration for $\pi < \theta_1 < 2\pi$ and these ranges are reversed for the positive solution.

3.1. Transmission ratio

Taking the derivative of the kinematic constraint equation with respect to θ_1 results in:

$$\frac{df}{d\theta_1} = 2ad\sin\theta_1 - 2cd\sin\theta_2\frac{d\theta_2}{d\theta_1} + 2ac\sin(\theta_1 - \theta_2)\left(1 - \frac{d\theta_2}{d\theta_1}\right) = 0 \tag{11}$$

which can be simplified to give the transmission ratio of the four-bar linkage, r_f.

$$r_f = \frac{d\theta_2}{d\theta_1} = \frac{\omega_2}{\omega_1} = \frac{ad\ \sin(\theta_1) + ac\sin(\theta_1 - \theta_2)}{cd\sin(\theta_2) + ac\sin(\theta_1 - \theta_2)} \tag{12}$$

For the parallelogram configuration of the mechanism, i.e $a = c$ and $b = d$, the expression for the transmission ratio simplifies to:

$$r_f = \frac{b/a\sin(\theta_1) + \sin(\theta_1 - \theta_2)}{b/a\sin(\theta_2) + \sin(\theta_1 - \theta_2)} \tag{13}$$

substituting a for c and b for d in A_1, A_2 and A_3 and simplifying Eq. (10) results in:

$$\tan\left(\frac{\theta_2}{2}\right) = \left(\frac{1 \pm b/a}{1 - b/a}\right)\tan\left(\frac{\theta_1}{2}\right) \tag{14}$$

obviously $\theta_2 = \theta_1$ will always be the case for the negative solution which corresponds to the parallelogram configuration (range $0 < \theta_1 < \pi$). Substituting this into the expression for the transmission ratio, r_f, reduced to unity for all input angles. For the crossed four-bar linkage configuration (positive solution for $0 < \theta_1 < \pi$), it can be proven using L'Hopital's rule, that $\theta_2 = -\theta_1$ only as the ratio b/a goes to infinity. Substituting $\theta_2 = -\theta_1$ into Eq. (13) and once more using L'Hopital's rule for b/a going to infinity, the transmission ratio, r_f, goes to −1.

Fig. 4 shows a plot of the parallelogram transmission ratio for the two solutions as a function of input angle, θ_1, and b/a ratio. Link a is given a length of 1 and the length of link b is varied starting at a length of 1 in steps of 1 to a length of 10 (notation: b = 1:1:10). Clearly, the negative solution (solid black lines) corresponds to the parallelogram in the range $0 < \theta_1 < \pi$ (i.e. $r_f = 1$ and it is independent of the ratio b/a) and the crossed configuration in the range $\pi < \theta_1 < 2\pi$.

It should be noted that a parallelogram four-bar can switch into the crossed configuration and vice versa at 0°, 180°, 360°, . . . (i.e. whenever all four links are aligned as the input and output link pass through 0° or 180°). However, assuming no switching between the two possible linkage configurations occurs, the input and output as well as coupler and fixed links remain parallel to each other—allowing synchronous motion to be achieved over the entire angular range of the input link. However, slight variation in the length of the links will make this impossible as is discussed Table 1.

3.2. Transmission ratio sensitivity

The sensitivity of the transmission ratio to errors in the link lengths can be determined by taking its partial derivate with respect to each of the link lengths. If the equation should not be readily differentiable, the sensitivity can be calculated numerically

Table 1
Effect of change in linkage type caused by small link length variations with the original linkage being a parallelogram with $a = c$, $b = d$ and $b > a$, a rocker only does intermittent motion while a crank can rotate 360°.

Link	Increase in link length	Decrease in link length
a	Rocker (input), crank (output): no continuous, 360° rotation of input link possible.	Crank (input), rocker (output): no continuous, 360° rotation of output link possible.
b	Double rocker	Double rocker
c	Crank (input), rocker (output): no continuous, 360° rotation of output link possible	Rocker (input), crank (output): no continuous, 360° rotation of input link possible.
d	Double rocker	Double rocker

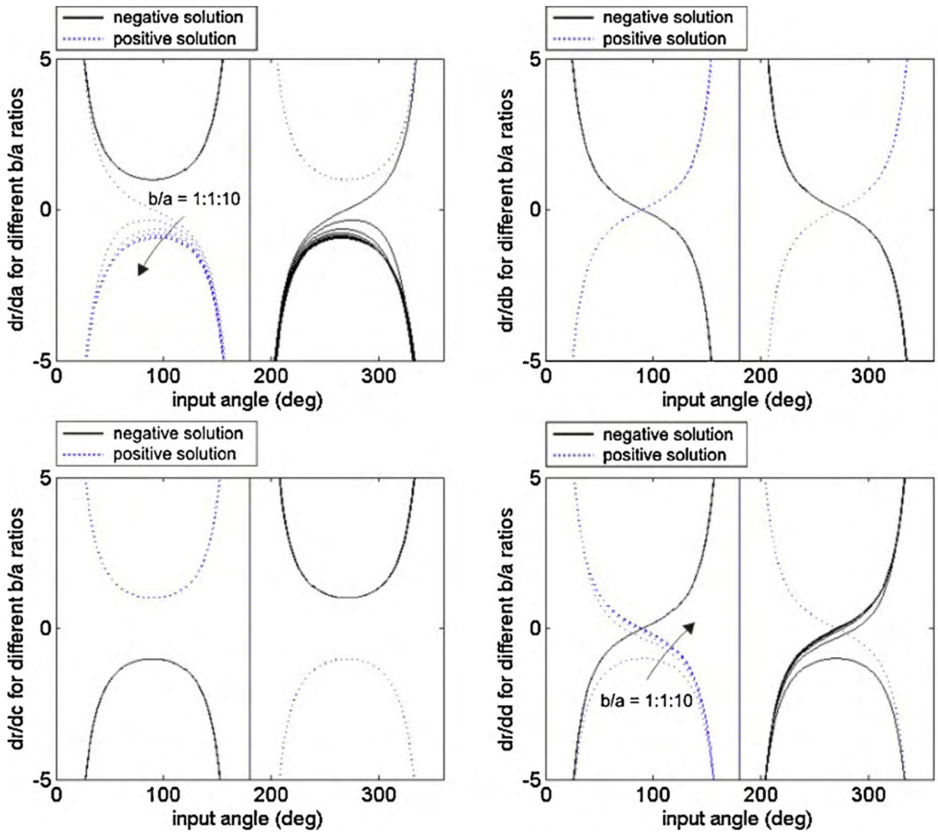

Fig. 5. Sensitivities of the transmission ratio when $a = c$ and $b = d$ to changes in the length of links a, b, c and d as a function of the ratio, b/a, and the input angle, θ_1. Singularities exist at 0° and 180°. The solid black lines corresponds to the negative solution (parallelogram in the range $0 < \theta_1 < \pi$ and the crossed configuration in the range $\pi < \theta_1 < 2\pi$).

as shown in Eq. (13) where Δ is a very small deviation in a link length (∼0.0001).

$$\frac{\partial r_f}{\partial a} = \frac{r_f(a+\Delta, b, \theta_1) - r_f(a-\Delta, b, \theta_1)}{2\Delta} \tag{15}$$

In Fig. 5, the sensitivities of the transmission ratio for the parallelogram configuration (Eq. (15)) to changes in the lengths of link a, b, c and d are shown as a function of input angle and the ratio of b/a. Again, the solid black lines corresponds to the negative solution (parallelogram in the range $0 < \theta_1 < \pi$ and the crossed configuration in the range $\pi < \theta_1 < 2\pi$).

4. Slider-crank

The slider-crank is a common linkage that is used to convert rotary motion to motion along a predefined guide-curve or vice versa. The analysis presented in this paper will be limited to slider-cranks with either a circular or a straight guide-curve (track), also called circular or straight slider-crank. However, the presented basic analysis technique can easily be generalized to any other given guide-curve.

4.1. Straight slider-crank

An illustration of a slider-crank with the end-point movement constrained to be along a linear track of slope m is shown in Fig. 6.

The mobility of the mechanism is 1 as the slider guide-way imposes a constraint on the system that otherwise would be a two degree of freedom two-link-manipulator. The kinematic constraint equation can be derived to be:

$$[a_s(\sin(\theta_1) - m\cos(\theta_1)) - y_0] - b_s m\cos(\theta_3) + b_s\sin(\theta_3) = 0 \tag{16}$$

again, as with the four-bar linkage constraint equation, the output angle, θ_3, can be found as a function of link lengths and input angle by solving:

$$B_1 + B_2\cos(\theta_3) + B_3\sin(\theta_3) = 0 \tag{17}$$

where

$$B_1 = -y_0 + a_s\,(\sin(\theta_1) - m\cos(\theta_1)) \tag{18}$$

$$B_2 = -b_s m \tag{19}$$

$$B_3 = b_s \tag{20}$$

Similar to the four-bar linkage, there are two solutions corresponding to the two possible configurations of the slider-crank linkage.

$$\tan\left(\frac{\theta_3}{2}\right) = \frac{-B_3 \pm \sqrt{{B_3}^2 - {B_1}^2 + {B_2}^2}}{B_1 - B_2} \tag{21}$$

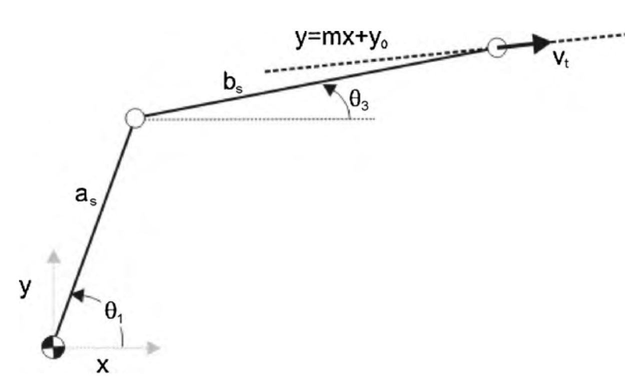

Fig. 6. Slider-crank with end-point riding on a straight line.

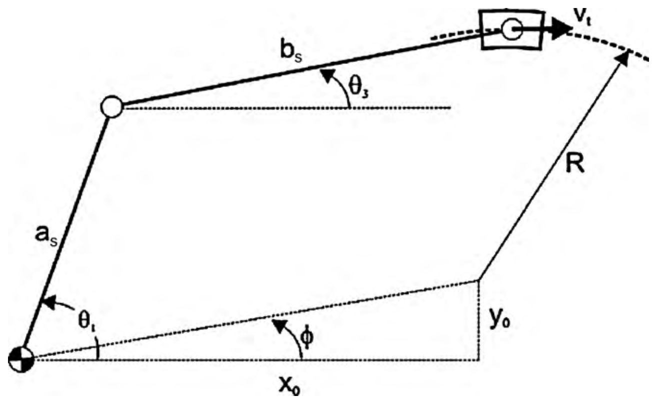

Fig. 7. Slider-crank with end-point riding on a circular path.

Differentiating the kinematic constraint equation for a straight slider-crank, Eq. (16), the transmission ratio between the output and input angles reduces to:

$$r_{\theta_1\theta_3} = \frac{d\theta_3}{d\theta_1} = \frac{\omega_3}{\omega_1} = \frac{a_s(\cos(\theta_1) + m\sin(\theta_1))}{b_s(\cos(\theta_3) + m\sin(\theta_3))} \quad (22)$$

The transmission ratio relating the output tangential velocity along the straight path, v_t, to the input angular velocity, $\omega_{1,}$ is

$$r_{ss} = \frac{v_t}{\omega_1} = \sqrt{a_s^2 + r_{\theta_1\theta_2}^2 b_s^2 + r_{\theta_1\theta_2} b_s\ \cos(\theta_1 - \theta_3)} \quad (23)$$

4.2. Circular slider-crank

As was mentioned earlier, another variation of a slider-crank is for the case when the end-point of the link is constrained to ride along a circular path, Fig. 7.

For this case, the kinematic constraint equation can be derived to be:

$$a_s^2 + b_s^2 + x_0^2 + y_0^2 - R^2 + 2a_s b_s\ \cos(\theta_1 - \theta_3)\ldots\ldots$$
$$-\ 2\left[x_0(a_s\cos\theta_1 + b_s\cos\theta_3) + y_0(a_s\sin\theta_1 + b_s\sin\theta_3)\right] = 0 \quad (24)$$

Again, the output angle, θ_3, can be found as a function of the input angle, θ_1, by solving Eq. (17) but now with the following values for the coefficients:

$$B_1 = a_s^2 + b_s^2 + x_0^2 + y_0^2 - R^2 - 2y_0 a_s\ \sin(\theta_1) - 2x_0 a_s\cos(\theta_1) \quad (25)$$

$$B_2 = 2a_s b_s\ \cos(\theta_1) - 2x_0 b_s \quad (26)$$

$$B_3 = 2a_s b_s \sin(\theta_1) - 2y_0 b_s \quad (27)$$

with x_t being the position of the slider end and v_t its velocity tangential to the guide-curve, the transmission ratio for the slider-crank with a circular guide-curve can be derived from its kinematic constraint equation to be:

$$r_{sc} = \frac{dx_t}{d\theta_1} = \frac{v_t}{\omega_1} = \frac{a_s R\ \sin(\theta_1 - \theta_3)}{(x_0{}^2 + y_0{}^2)^{1/2}\sin(\theta_3 - \varphi) + a_s\sin(\theta_1 - \theta_3)} \quad (28)$$

It should be noted that the circular slider-crank corresponds to a four-bar linkage with the constraint of the *c* link replaced by a circular guide-way. Consequently, in the case of a circular slider-crank, it is possible to achieve a constant transmission ratio, r_{sc} (equivalent to a parallelogram four-bar). However, the straight slider-crank does not have any constant transmission ratio. The transmission ratio sensitivity of a slider-crank can be analyzed using the same mathematical method that was demonstrated for the parallelogram four-bar linkage.

It should be noted that due to typically wider manufacturing tolerances/errors in prismatic joints (or sliders) compared to pivots of similar cost, it is advisable to avoid designs with circular slider-cranks unless space constraints prevent the use of a functionally equivalent four-bar linkage.

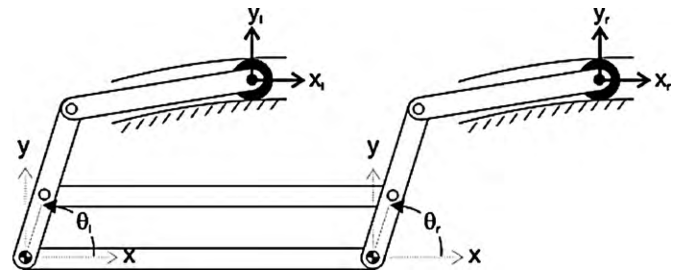

Fig. 8. Synchronous motion mechanism where a four-bar linkage couples the motion of two slider-cranks.

5. Case study

5.1. Synchronous motion mechanism

As an example, consider the mechanism in Fig. 8 that is designed to produce synchronous motion between the end-points of two links riding in circular tracks. It consists of two identical slider-cranks that are coupled by a parallelogram four-bar linkage.

An actuator is assumed to drive the left input angle, θ_l, of the four-bar linkage, to which the left slider-crank is directly coupled. It is assumed that the range of motion for the input link is $50^\circ \le \theta_l \le 130^\circ$, thus not passing through 0° or 180° so as to avoid singularities. The right side slider-crank is directly coupled to the output link of the four-bar linkage. Should it be possible to ensure that the links are manufactured precisely and sufficiently stiff so as to not significantly deform under any applied load, smaller than the stall torque of the motor, then the right slider end-point would exactly track the left slider end-point, i.e. there would be zero following error and the end-points would move in synchrony. The end-points of the left and right slider-cranks are (x_l, y_l) and (x_r, y_r), respectively.

5.2. Link length variation

In order to simulate the effect of variations in link lengths on the transmission ratios and following errors of the mechanism, the parameters of the mechanism illustrated in Fig. 8 are given the values shown in Table 2. The labeling for the four-bar and slider-cranks corresponds to that used in Figs. 3 and 7.

The values in Table 2 indicate that the four-bar linkage is in the parallelogram configuration and thus with a nominal angular transmission ratio of 1. The *b*/*a* ratio was chosen to be 10 to mimic a typical application where a linkage is used to transmit motion over a long distance. However, let us now consider the case when link *b* is 1% longer than specified, causing asymmetry and a deviation from the desired parallelogram configuration of the four-bar linkage. For simplicity, we will assume that there are no dimensional errors for all other link lengths of the four-bar linkage and slider-crank

Table 2
Parameter values for four-bar linkage and slider-crank for case study.

Four-bar linkage		Slider-cranks	
a	1	a_s	2
b	10	b_s	5
c	1	x_0	3
d	10	y_0	0.25
		R	4

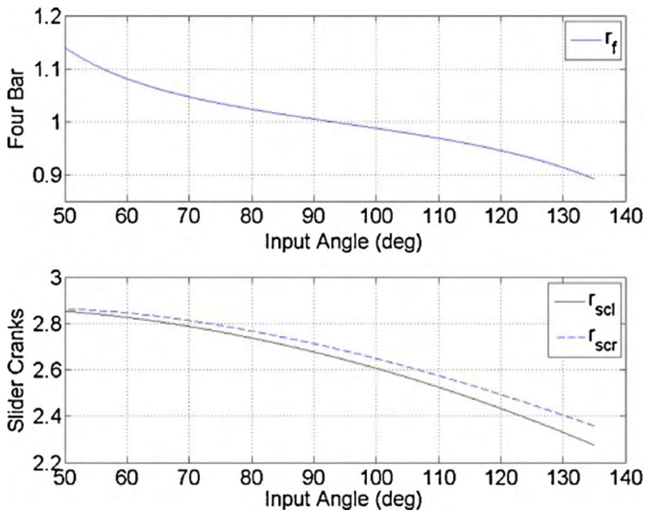

Fig. 9. Transmission ratios for four-bar linkage, r_f (top), left circular slider-crank and (four-bar × right circular slider-crank), r_{scl} and r_{scr} (bottom).

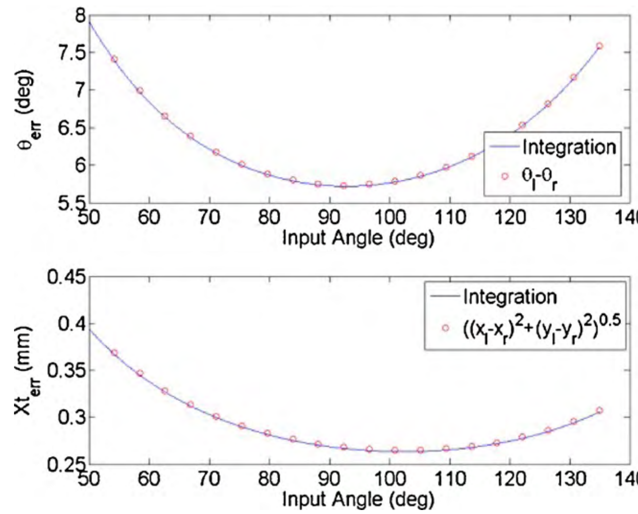

Fig. 10. Plot of the angular and end-point following error comparing the values obtained using Eqs. (30) and (32) with Eqs. (33) and (34), respectively. As expected, the curves methods match identically.

mechanisms. The effect of the error in the length of link b on the transmission ratios is visualized in Fig. 9.

The four-bar linkage transmission ratio deviates significantly from the ideal value of 1 over the input range ($-10\% \le r \le +20\%$). The effect of this is to also cause a difference in the slider-crank transmission ratios for a given input angle, θ_l. This is because the output angle of the four-bar linkage no longer equals the input angle (the output angle of the four-bar linkage is the input angle of the right slider-crank).

5.3. Following error

This difference between the input and output angle of the four-bar linkage (defined as the angular following error) can be calculated using the transmission ratio. The incremental angular following error, $\delta\theta_{err}$, can be defined in terms of the incremental angular input, $\delta\theta_l$, and the four-bar linkage transmission ratio, r_f.

$$\delta\theta_{err} = \delta\theta_l - \delta\theta_r = (1 - r_f)\delta\theta_l \tag{29}$$

This incremental error can then be integrated over the input range to find the total angular following error, θ_{err}.

$$\theta_{err} = \int_{\theta_{l,\mathrm{start}}}^{\theta_{l,\mathrm{end}}} (1 - r_f)\delta\theta_l \tag{30}$$

Further, the transmission ratios can be combined in the case of coupled mechanisms to find the overall following error without having to compute the transmission ratio for the entire mechanism. For this mechanism, the total end-point following error of the complete mechanism is defined as the tangential difference between the end positions of the two slider-cranks, Xt_{err}. Its incremental value can be calculated in a similar manner to the angular following error.

$$\delta Xt_{err} = \delta Xt_l - \delta Xt_r = \delta\theta_l r_{scl} - \delta\theta_r r_{scr} = (r_{scl} - r_f r_{scr})\delta\theta_l \tag{31}$$

which can be integrated over the input range to find the total following error between the end-points of the slider-cranks

$$Xt_{err} = \int_{\theta_{l,\mathrm{start}}}^{\theta_{l,\mathrm{end}}} (r_{scl} - r_f r_{scr})\delta\theta_l \tag{32}$$

The above equations illustrate that the transmission ratio of a mechanism can be used to calculate the following error. For the simple case study presented in this paper, the angular and end-point following errors of the synchronous motion mechanism can easily be calculated from Eqs. (33) and (34)

$$\theta_{err} = \theta_l - \theta_r \tag{33}$$

$$Xt_{err} = \sqrt{(x_l - x_r)^2 + (y_l - y_r)^2} \tag{34}$$

Fig. 10 plots the following errors using the two methods described. For ease of comparison and clarity, the following errors from Eqs. (33) and (34) are plotted only at discrete points. However, the curves are identical over the entire input range as expected.

6. Conclusions

Based on the example of four-bar linkages and slider-crank mechanisms, a method was presented to derive the transmission ratio of a single degree of freedom mechanism from its respective kinematic constraint equation. The transmission ratio, along with its sensitivity to geometrical variations, can then be used to predict the performance for a given mechanism design.

It was shown that a constant transmission ratio of unity can be achieved by a parallelogram four-bar linkage or its equivalent circular slider-crank. Reversal of direction of rotation between input and output can be achieved in the corresponding crossed configuration of the four-bar which also has an equivalent slider-crank. However, a constant transmission ratio of -1 over 360° of input motion can only be achieved for a coupler-input link length ratio of infinity and thus is practically not feasible. Furthermore, for a parallelogram four-bar linkage, it was shown that a transmission ratio of 1 is highly sensitive to variations in the lengths of the links and thus care must be taken to understand how it deviates from unity as a function of dimensional tolerances, operating range, coupler-input link length ratio and mechanism configuration. E.g. for the parallelogram configuration with a link length ratio b/a of ten, 1% variation in the length of link b results in an angular following error between -11% and 14% over an input range of motion of 80°.

Based on a case study of a synchronous motion mechanism consisting of two slider-cranks that are coupled by a four-bar linkage, it was demonstrated how small (undesired) deviations in the rocker link length, and subsequently the transmission ratio, can cause significant following errors between the two ideally synchronized outputs of the mechanism.

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Appendix A. Link forces in four-bar linkage and slider-crank

A.1. Static equilibrium

Newton's first two laws of motion are that, if a body is at rest, the sum of all forces acting on the body must be zero. Further, the sum of the moments of those forces about any point must also be zero. Thus for any member of the structure

$$\sum F_x = 0 \text{ and } \sum F_y = 0 \tag{A1.1}$$

$$\sum M_o = 0 \tag{A1.2}$$

where $\sum F_x$ and $\sum F_y$ are the sum of all forces acting on the body in the x and y directions, respectively (with respect to the reference coordinate system) and $\sum M_o$ is the sum of the moments of those forces about any chosen point, o. The basis of the static force analysis of any structure is the algebraic solution of the static equilibrium equations written for every member of the system.

A.2. Free body diagrams

The forces in each of the links of the four-bar mechanism can be calculated based on a static analysis of the mechanism. The usual approach to solving this problem is to sketch a free body diagram for each member of the mechanism. All forces acting on each member, including the forces of action and reaction between members, as well as externally applied loads must be indicated on the free body diagram. The forces that can be transmitted across an ideal (frictionless) kinematic joint are related to the motions permitted by that joint. Basically, the work done by the transmitted forces in the directions of permitted motion must be zero. For example, a revolute joint permits rotation about its axis. Any force that is normal to that axis and whose line of action intersects it does no rotational work. Therefore, any force component in the plane of motion passing through the joint axis is transmitted. It is usually convenient to represent this set of possible forces by two components parallel to the x and y axis directions of the fixed reference frame.

A.3. Force analysis

The force F_{ab} is interpreted as the force that link a exerts on link b and force F_{ba} is the force that link b exerts link a and these are obviously equal and opposite as discussed above. If a force is not an internal force between two rigid bodies, the subscript will correspond to the location where the force is applied or to the type of force. For a force analysis, we must know the coordinates of all points involved in the analysis. Therefore, before a force analysis can be conducted a positional analysis must be done as described earlier. The position equations will be nonlinear but the equations for the forces will be linear and can easily be solved as is outlined here.

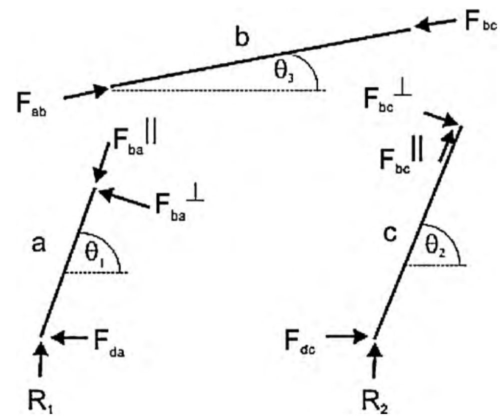

Fig. 11. Free body diagram for four-bar linkage.

A.4. Four-bar linkage

The free body diagram of a four-bar mechanism is shown in Fig. 11. The input torque is applied to link a, e.g. by a motor.

Applying force and moment balance to each of the links in the free body diagram allows each of the forces in the four-bar mechanism to be calculated. The output torque that is calculated is the input torque that is applied to the secondary slider-crank mechanism. The equations are as follows.

$$F_{ba} = \frac{\tau}{a \cdot \cos(\theta_3 + 90 - \theta_1)} \tag{A1.3}$$

$$F_{da} = -F_{ba} \cdot \cos(\theta_3) \tag{A1.4}$$

$$R_1 = F_{ba} \cdot \sin(\theta_3) \tag{A1.5}$$

$$F_{ab} = -F_{ba} \tag{A1.6}$$

$$F_{cb} = F_{ab} \tag{A1.7}$$

$$F_{bc} = -F_{cb} \tag{A1.8}$$

$$F_{dc} = -F_{bc} \cdot \cos(\theta_3) \tag{A1.9}$$

$$R_2 = -F_{bc} \cdot \sin(\theta_3) \tag{A1.10}$$

$$\tau_{\text{out}} = F_{bc} \cdot c \cdot \cos(\theta_3 - 90 - \theta_2) \tag{A1.11}$$

These equations also allow the force in each link to be calculated for any configuration of the mechanism. The stiffness of the links calculated using finite element analysis. Combining these with the forces allows the deflections (and resulting errors) of the various links of the mechanism to be calculated.

A.5. Slider-crank

The free body diagram of the slider mechanism is shown in Fig. 12. The input torque is applied to link a_s e.g. by a motor. The output force is calculated.

In general, the forces can only be applied in the direction of the links. Therefore:

$$R = F \tag{A1.12}$$

$$T = F \begin{bmatrix} \cos(\theta_3) \\ \sin(\theta_3) \\ 0 \end{bmatrix} \times \begin{bmatrix} a\cos(\theta_1) \\ a\sin(\theta_1) \\ 0 \end{bmatrix}$$

The forces tangential and perpendicular to the curve are given by:

$$\begin{aligned} F_{\|} &= F\ \cos(\alpha) \\ F_{\perp} &= F \sin(\alpha) \end{aligned} \tag{A1.13}$$

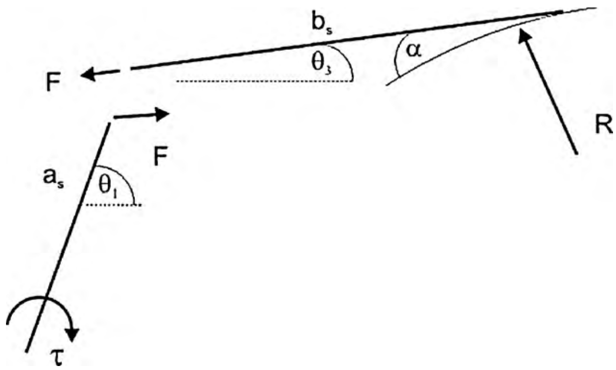

Fig. 12. Free body diagram for slider-crank.

Which can be expressed as:

$$F_{||} = \frac{T\ \cos(\alpha)}{a_s \sin(\theta_1 - \theta_3)}$$
$$F_{\perp} = \frac{T \sin(\alpha)}{a_s \sin(\theta_1 - \theta_3)} \quad \text{(A1.14)}$$

Finally, the frictional force can be obtained from:

$$F_{\text{friction}} = \mu F_{\perp} = \mu \frac{T\ \sin(\alpha)}{a_s \sin(\theta_1 - \theta_3)} \quad \text{(A1.15)}$$

A.6. A note on load-induced deflection

So far we have only discussed the effect of errors in the link lengths on the transmission ratio and following errors. While there will always be some variation as a result of manufacturing tolerances, links may also deviate from there nominal length as a result of loads on the mechanism. However, the difficulty in modeling load-induced errors lies in their often distributed and/or varying effects. They are often distributed throughout the structure, and thus in order to calculate the resulting errors in a mechanism, a method for lumping them at discrete points must be devised. Bearing interfaces are often chosen as the location to lump load-induced errors because they define the kinematics of the mechanism and are often the most compliant parts of the mechanism. Appendix A outlines the methods to calculate the link forces for the four-bar linkage and slider-crank. In general, the forces depend on the configuration of the mechanism and so a position analysis must be performed first. In a next step the forces parallel and perpendicular to each link have to be calculated. This should be an easy task once the forces in the x and y directions have been obtained. The stiffness of the individual components can then be computed by means of beam theory for simple parts or finite element analysis for parts with more complicated geometry.

The procedure for determining the following error between the end-points of the two slider-cranks due to load-induced deformation in the above case study is as follows:

1. Set the initial input angle of the four-bar linkage.
2. Find the change in this angle due to bending of link a and add it to the input angle to get a new input angle.
3. Find the change in all of the link lengths due to the force components along their lengths.
4. Find the new output angle based on change in input angle and link lengths.
5. Find the change in the output angle due to bending of link c and add it to get a new output angle.
6. This is the new input angle to the secondary slider-crank.
7. Find the change in the slider-crank input angle due to bending of link a_s of the slider-crank and add it to the input angle to get a new input angle.
8. Find the change in all of the link lengths due to the force components along their length.
9. Find the new configuration of the slider-crank based on change in input angle and link lengths.
10. Find the new end position of the slider-crank.
11. Calculate the error between this new position and the position calculated from the kinematics originally.
12. Recalculate the forces with using the deformed link lengths.
13. Compare the forces as calculated in step 12 to the originally calculated forces. This will give you an error measure for the accuracy of your analysis—remember forces are configuration dependent and thus forces might change as the mechanism deforms.

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